On Equivalent Resistance of Electrical Circuits

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Abstract

One of the basic tasks related to electrical circuits is computing equivalent resistance. In some simple cases, this task can be handled by combining resistors connected either in series or in parallel, until the original circuit reduces to a single element. When this is not possible, one resorts to the "heavy artillery" of Kirchhoff′s rules or method of nodal potentials. In this paper, we apply the latter method to derive –in a closed form– the equivalent resistance of a generic circuit. This result unveils a curious interplay between electrical circuits, matrix algebra, graph theory and its applications to computer science.

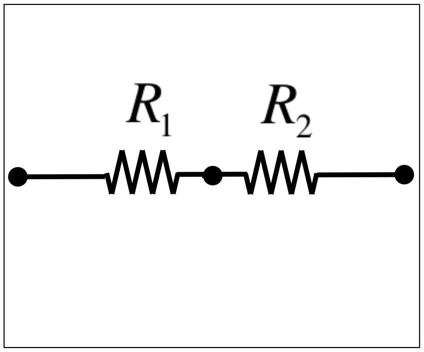
Introduction

From the mathematical perspective, a resistive electric circuit can be understood as a graph whose edges are assigned given numerical values of resistance (R). In addition, two nodes are assumed to be "connected to the battery'', which fixes the *potential difference* (voltage, $\mathcal{E}$) between the two nodes. The battery gives rise to currents flowing along the edges of the circuit. These edge currents (I) are related to the potential differences across the corresponding nodes (ΔV) via Ohm's Law:

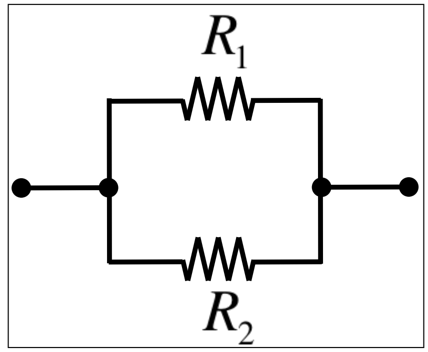
$$I = \frac{\Delta V}{R}. \tag{1}$$

The total current flowing through all the edges coming out of a battery node is called the output current (I_{out}). The total current that flows into the other battery node is equal to the output current. One of the important characteristics of an electrical circuit is its *equivalent resistance* (R_{eq}). It is defined as the resistance of a single resistor (edge) that, if it were to replace the whole circuit, would result in the same amount of the output current. In other words

$$R_{\text{eq}} := \frac{\mathcal{E}}{I_{\text{out}}}. \tag{2}$$



(a) Connection in Series



(b) Connection in Parallel

Figure 1: Simplest connections of resistors

Although the output current is explicitly present in the definition of equivalent resistance, it is clear (e.g. on the dimensional grounds) that R_{eq} depends only on the given edge resistances. In some simple cases, it can be computed without finding the output current. For instance, if two edges are connected via a two-valent node (Fig.1a), this is referred to as a connection *in series*, and the individual resistances are merely added up. If two resistors are on edges connected across the same pair of nodes (Fig.1b), such a connection is called *in parallel*, and the equivalent resistance is computed as the reciprocal of the sum of reciprocals. Both of these situations can be generalized straighforwardly for more than two resistors in series or parallel respectively:

$$R_{\text{eq}} = R_1 + R_2 + \ldots, \qquad \text{for connection in series} \tag{3}$$

$$R_{\text{eq}} = \left(R_1^{-1} + R_2^{-1} + \ldots\right)^{-1}, \qquad \text{for connection in parallel} \tag{4}$$

Some larger circuits allow for reduction by identifying subcircuits whose elements are connected either in series or in parallel. Replacing such subcircuits by single equivalent resistances turn by turn may result in a trivial circuit thus giving an algorithm for computing equivalent resistance.

At the same time, it is clear that not every circuit can be simplified this way. For example, a circuit with no parallel edges and whose nodes are at least three-valent (such nodes are referred to as *junctions*) cannot be reduced in the above sense. The simplest non-simplifiable circuit is depicted in Fig. 2. It is easy to see that there are no elements connected in series or parallel. In

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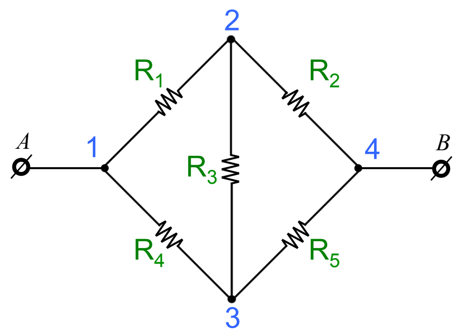

Figure 2: The simplest non-simplifiable circuit (*a-la* Wheatstone bridge). There are no connections in series or in parallel.

order to determine the equivalent resistance of such a circuit, one typically introduces the unknown edge currents and writes down Kirchhoff's rules. After solving those equations for the unknown currents, one can compute the output current and obtain the equivalent resistance from (2).

Kirchhoff's rules come in two types: junction rules and loop rules, which follow from the charge and energy conservation equations respectively.

- The total current that goes into a junction equals the total current that comes out of it. Equivalently, the algebraic sum of all the currents for every junction equals zero, assuming that incoming and outgoing currents are assigned oppposite signs. This, of course, holds only for junctions that are not the battery terminals.

- For every closed loop in the circuit, the resultant potential difference equals zero. When traversing the chosen loop, the contribution to the overall potential difference comes from e.g. resistors carrying current (using Ohm's Law (1)), batteries etc.

Some of the resulting Kirchhoff's equations are not independent, but the number of the independent ones, including both junction and loop rules, is precisely equal to the number of the unknown edge currents.

There is also an alternative approach to finding unknowns quantities associated with electrical circuits, called the *method of nodal potentials* (see e.g. [1]). Here basic variables are the potentials at the circuit's nodes rather than the edge currents. The advantage of this method is that it usually deals with (much) fewer variables than in Kirchhoff's framework. In addition, unlike Kirchhoff's rules, the equations for the unknown nodal potentials are of one type. This presents a major simplification for analytical description of electrical circuits, as well as for finding various quantities of interest.

In this paper, we start by reviewing the method of nodal potentials for electrical circuits, highlight some immediate implications of the formalism and work out a sample circuit (Wheatstone brindge). For generic values of edge resistance, the latter is somewhat non-trivial when applying Kirchhoff's rules, but is quite straightforward using the method of nodal potentials. In section 2, we then proceed to deriving a closed formula for the equivalent resistance/conductance of an arbitrary resistive circuit (see Eq.(19)). In the last section, we discuss generalizations of the formula, as well as its possible relation to matrix algebra, graph theory and its applications to computer science.

1 Method of nodal potentials

In this section we consider a convenient description of resistive DC-circuits containing one battery and investigate its implications. We shall see how the method of nodal potentials can simplify the analysis of an electric circuit compared to the standard Kirchhoff's rules technique. The same approach can be applied to DC-circuits with several batteries, as well as to generic AC-circuits. Such generalizations are discussed in the last section.

1.1 Formalism and notation

Consider a circuit containing n nodes, such that nodes 1 and n are connected to the positive and negative terminals of the battery respectively (Fig. 3). Every link connecting two nodes i and j is assumed to have known resistance $R_{ij} \equiv R_{ji}$. In fact from now on, it will be more convenient to use the conductance rather than resistance, defined as

$$\sigma_{ij} := \frac{1}{R_{ij}},$$

thus giving rise to the conductance matrix $\Sigma := \sigma_{ij}$. It is easy to see that the rules for computing equivalent conductance are reversed compared to those for equivalent resistance:

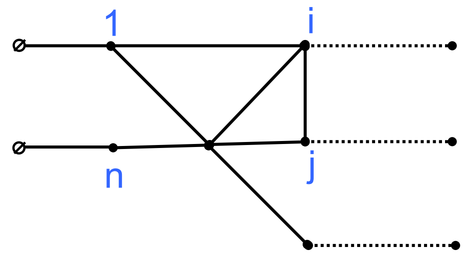

Figure 3: A generic circuit with n nodes. The first and last nodes are connected to the battery terminals.

$$\sigma_{\text{eq}} = \sigma_A + \sigma_B + \ldots, \qquad \text{for connection in parallel} \tag{5}$$

$$\sigma_{\text{eq}} = \left(\sigma_A^{-1} + \sigma_B^{-1} + \ldots\right)^{-1}, \qquad \text{for connection in series} \tag{6}$$

Without loss of generality, we can make the following assumptions:

- $V_1 = \mathcal{E}$ and $V_n = 0$. Since electric potential is defined up to a constant, we fix one of them to zero. The potential at the positive terminal, $\mathcal{E}$, can be used as a unit.

- Every node is connected to every other node. If, in reality, some nodes i, j etc. do not share a link, we simply put $\sigma_{i,j} = 0$.

The latter assumtion will allow us to not worry about the circuit's topology and concentrate on purely algebraic description. In fact, if we only focus on non-simplifiable ciruits, every node, except possibly the 1$^{\text{st}}$ and n^{th} ones, will have at least three edges. Indeed, a node with only two edges would imply a connection of edges in series which could be replaced by a single edge. In Appendix B, we discuss how this and other simplifications affect the Σ-matrix. Note that each column/row of the Σ-matrix for a non-simplifiable circuit must have at least three non-zero entries.

We now define the edge current I_{ij} as the current flowing from node i to node j. Its expression can be readily written in terms of the nodal potentials as

$$I_{ij} := \sigma_{ij} \left(V_i - V_j\right) \equiv -I_{ji}, \tag{7}$$

which is manifestly anti-symmetric. Since electric current flows from a higher potential to a lower one, this definition sets an outgoing current to be positive, whereas an incoming current would be negative.

We conclude this subsection with the following remark. A minimally connected, non-simplifiable circuit has at least $\sim 3n/2$ edges, hence the same number of unknown currents. At the same time, the number of unknown nodal potentials is $(n-2)$, which can be substantially less than the number of currents. Thus the method of nodal potentials deals with fewer variables at the on-set.

1.2 Implications

Assuming that the variables describe a real circuit which has specific unique values of the nodal potentials, the loop rules will be automatically satisfied by construction: any closed loop will come back to the same value of potential, hence making the overall potential difference zero.

For a generic circuit, the junction rules take the form

$$\sum_{j=1}^{n} I_{ij} \equiv \sum_{j=1}^{n} (V_i - V_j)\, \sigma_{ij} = 0, \quad \text{for } i \neq 1 \text{ or } n. \tag{8}$$

The immediate implication of these equations is the following

Lemma 1 *The total current flowing into the* n^{th} *node is equal to the total current flowing out of the* 1^{st} *node.*

Proof: Since the edge currents are anti-symmetric, $I_{ij} = -I_{ji}$, the sum over both indices $\sum_{i,j=1}^{n} I_{ij} = 0$. Splitting the summation over i into $i = 1$, $i = n$ and the rest yields

$$\sum_{j=1}^{n} I_{1,j} + \sum_{j=1}^{n} I_{n,j} + \sum_{i=2}^{n-1} \left(\sum_{j=1}^{n} I_{ij} \right) = 0.$$

By virtue of Eq.(8), each summand in the last term vanishes. The first and second terms are the output and (negative) input currents respectively, which proves the lemma. $\square$

While the statement of the lemma was not mathematically obvious from the construction, it makes clear physical sense: since there is no accumulation of charge in the circuit, the total current coming out of one terminal of the battery has to equal the current flowing back into its other terminal.

There is another statement which is obvious from the physical point of view, but non-trivial mathematically[2]:

Lemma 2 *The values of the nodal potentials in a resistive circuit connected to a single battery should lie (strictly) between the lower battery voltage and the higher one, i.e. between* 0 *and* $\mathcal{E}$*. In other words, the battery sets the lowest and the highest possible potential in the circuit.*

Proof: Suppose that the maximum potential is attained at node $m \neq 1$. Then all the neighboring nodes would have a lower potential resulting in currents from node m

$$I_{mj} = \sigma_{mj} \left(V_m - V_j \right) > 0$$

to be outgoing. The latter would violate the junction rule (8). Thus the maximum potential can't be attained at any node other than the 1st one. Similarly, the minimum potential is attained at node n. Hence all the nodal potential values lie between V_1 and V_n, which proves the lemma. □

In the next session we illustrate how the method of nodal potentials helps to determine the equvalent conductance of the Wheatstone bridge circuit with arbitrary edge conductances.

1.3 An example: Wheatstone bridge circuit

We now assume that the values of edge resistance in Fig.2 are given and find the equivalent resistance (conductance) of the circuit. Notice that if one was to solve this problem using Kirchhoff's rules, one would have to deal with six unknown currents: one in each resistor plus the output current. At the same time, there are only two unknown nodal potentials: V_2 and V_3, since $V_1 = \mathcal{E}$ and $V_4 = 0$. For convenience, we also label the edge conductances similarly to the original resistances:

$$\sigma_{12} \equiv \sigma_1 = \frac{1}{R_1}, \quad \sigma_{24} \equiv \sigma_2 = \frac{1}{R_2}, \quad \sigma_{23} \equiv \sigma_3 = \frac{1}{R_3}, \quad \sigma_{13} \equiv \sigma_4 = \frac{1}{R_4}, \quad \sigma_{34} \equiv \sigma_5 = \frac{1}{R_5}.$$

It is easy to see that there are exactly two junction equations for this circuit, one for junction 2 and one for junction 3:

[2]Here we assume that all the relevant nodes are electrically connected to the battery and that no circuit node is connected to the baterry terminal via an ideal wire, i.e. via an edge of zero resistance.

$$I_{21} + I_{24} + I_{23} = \sigma_1 (V_2 - V_1) + \sigma_2 (V_2 - V_4) + \sigma_3 (V_2 - V_3) = 0 \tag{9}$$

$$I_{31} + I_{34} + I_{32} = \sigma_4 (V_3 - V_1) + \sigma_5 (V_3 - V_4) + \sigma_3 (V_3 - V_2) = 0 \tag{10}$$

Setting $V_1 = \mathcal{E}$ and $V_4 = 0$ and solving the equations on the righthand side for the unknown potentials V_2 and V_3, we obtain:

$$V_2 = \mathcal{E}\frac{\sigma_1\sigma_{345} + \sigma_3\sigma_4}{\sigma_{123}\sigma_{345} - \sigma_3^2}, \quad V_3 = \mathcal{E}\frac{\sigma_4\sigma_{123} + \sigma_1\sigma_3}{\sigma_{123}\sigma_{345} - \sigma_3^2}, \tag{11}$$

where $\sigma_{123} \equiv \sigma_1 + \sigma_2 + \sigma_3$ and $\sigma_{345} \equiv \sigma_3 + \sigma_4 + \sigma_5$.

It is important to understand that for a connected circuit ($0 < \sigma_{ij} < \infty$) the denominator in (11) can never be zero. To make it more transparent, we rewrite the denominator as

$$\mathcal{D} = \sigma_3(\sigma_1 + \sigma_2 + \sigma_4 + \sigma_5) + (\sigma_1 + \sigma_2)(\sigma_4 + \sigma_5).$$

Since each term in $\mathcal{D}$ is non-negative, it can only be zero if each term is zero. Irrespective of whether $\sigma_3 = 0$, the latter implies that $\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = 0$. This corresponds to a disconnected circuit, such that nodes 2 and 3 are completely isolated, which makes their potentials undetermined [3].

In order to find the equivalent capacitance, it is easiest to consider the current flowing into node 4, $I_{24} + I_{34} = \sigma_2 (V_2 - V_4) + \sigma_5 (V_3 - V_4)$. Setting $V_4 = 0$ and using the nodal potentials in (11), we obtain

$$\sigma_{\text{eq}} = \frac{\sigma_2 V_2 + \sigma_5 V_3}{\mathcal{E}} = \frac{\sigma_1\sigma_2\sigma_{345} + \sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_3\sigma_5 + \sigma_4\sigma_5\sigma_{123}}{\sigma_{123}\sigma_{345} - \sigma_3^2}. \tag{12}$$

Looking at the answer, we see that it is a ratio of two polynomials of degree $(n-1)$ and $(n-2)$ respectively. This clearly guarantees the correct units of conductance. Moreover, each polynomial is a sum of non-negative terms. As explained in the above footnote, this means that (11) and (12) are neither zero nor infinity for any connected circuit.

Finally, the Wheatstone bridge has a well known feature that for a special arrangement of edge resistances (conductances), there is no current through

[3] It could be the case that all conductances but, say, σ_1 were equal zero. Then on physical grounds, that would fix $V_2 = V_1 = \mathcal{E}$. Mathematically, it can be seen from (11) by giving σ_3 (which would not affect the circuit connectivity or V_2 anyway) a small non-zero value. The latter would result in cancellations and $V_2 = \mathcal{E}$.

the middle wire (labeled with σ_3). We can arrive this condition by setting equal potentials at the ends of the wire $V_2 = V_3$. This yields $\sigma_1\sigma_5 = \sigma_2\sigma_4$ or the standard

$$\frac{R_1}{R_2} = \frac{R_4}{R_5}. \tag{13}$$

In the next section we shall generalize the expression for the nodal potentials (11) in an arbitrary circuit.

1.4 Expressions for the nodal potentials in a generic circuit

We now revisit the generic circuit displayed in Fig. 3 with the same assumptions as in Sec. 1.1: the battery terminals read $V_1 = \mathcal{E}$ and $V_n = 0$ and all the edge conductances σ_{ij} are given. We can write the junction equations, analogous to (9) and (10), for all the nodal potentials, including V_1 and V_n. Collecting similar terms, we can state those equations in the following matrix form

$$\begin{pmatrix} c_1 & -\sigma_{12} & -\sigma_{13} & \dots & -\sigma_{1,n-1} & -\sigma_{1,n} \\ -\sigma_{21} & c_2 & -\sigma_{23} & \dots & -\sigma_{2,n-1} & -\sigma_{2,n} \\ -\sigma_{31} & -\sigma_{32} & c_3 & \dots & -\sigma_{3,n-1} & -\sigma_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\sigma_{n-1,1} & -\sigma_{n-1,2} & -\sigma_{n-1,3} & \dots & c_{n-1} & -\sigma_{n-1,n} \\ -\sigma_{n,1} & -\sigma_{n,2} & -\sigma_{n,3} & \dots & -\sigma_{n,n-1} & c_n \end{pmatrix} \begin{pmatrix} {\color{green}V_1} \\ {\color{red}V_2} \\ {\color{red}V_3} \\ \vdots \\ {\color{red}V_{n-1}} \\ {\color{green}V_n} \end{pmatrix} = \begin{pmatrix} {\color{red}I_1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ {\color{red}I_n} \end{pmatrix}, \tag{14}$$

where the green quantities are given, the red ones are unknown, and the diagonal elements $c_i = \sum_{j=1}^{n} \sigma_{ij}$. We denote this matrix Σ [4] and will also need Σ', its upper-left sub-matrix $(n-1) \times (n-1)$, as well as

$$\Sigma'' = \begin{pmatrix} c_2 & -\sigma_{23} & \dots & -\sigma_{2,n-1} \\ -\sigma_{32} & c_3 & \dots & -\sigma_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_{n-1,2} & -\sigma_{n-1,3} & \dots & c_{n-1} \end{pmatrix}, \tag{15}$$

[4] Interestingly, the matrix Σ is obtained from the conductance matrix σ_{ij} via the same procedure as obtaining the Laplacian matrix from the adjacency matrix. Namely, the off-diagonal elements of Σ are simply $-\sigma_{ij}$, whereas each diagonal entry is the sum of the elements of the corresponding row of σ_{ij}. Note also that the sum of all elements of Σ equals zero, hence at most $(n-1)$ equations in (14) are independent.

the lower-right sub-matrix of Σ' of size $(n-2)\times(n-2)$. The first element of each row of Σ multiplies $V_1 = \mathcal{E}$. Carrying this term to the righhand side of each equation in (14), we can obtain an explicit matrix equation for V_2, V_3, ..., V_{n-1}, with Σ'' being the relevant matrix on the lefthand side.

Applying Cramer's rule to rows 2 through $(n-1)$ in (14), we obtain the following expressions for the nodal potentials

$$V_i = \mathcal{E}\frac{\det \Sigma''_k}{\det \Sigma''}. \tag{16}$$

Here Σ''_k is the matrix obtained from Σ'' by substituting $(\sigma_{21}, \sigma_{31}, ..., \sigma_{n-1,1})^{\mathrm{T}}$ instread of its k^{th} column. On physical grounds, for any connected circuit the determinant in the denominator should be non-zero. This, however, is not so obvious from the mathematical point of view. See Appendix A for more detail.

2 Derivation of the equivalent conductance of a generic circuit

In principle, one can find the equivalent conductance of the ciruit in Fig.3 using the same method as in Sec. 1.3. Specifically, we could use the nodal potentials found in (16) to compute the output current and substitute in

$$\sigma_{\text{eq}} = \frac{I_{\text{out}}}{\mathcal{E}}.$$

There is, however, a more economical way to arrive at the expression for the equivalent conductance. Interestingly, it can be obtained in a closed form. Moreover the expression of σ_{eq} in terms of the individual conductancies (which are assumed given) is universal and does not require prior finding nodal potentials. The derivation is as follows.

We first rearrange terms in Eqs. (14), so that all the unknowns are on the lefhand side and all the givens are on the righthand side of the each equation

$$\begin{pmatrix} -1 & -\sigma_{12} & -\sigma_{13} & \dots & -\sigma_{1n-1} \\ 0 & c_2 & -\sigma_{23} & \dots & -\sigma_{2n-1} \\ 0 & -\sigma_{32} & c_3 & \dots & -\sigma_{3n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\sigma_{n-12} & -\sigma_{n-13} & \dots & c_{n-1} \end{pmatrix} \begin{pmatrix} I_{\text{out}} \\ V_2 \\ V_3 \\ \vdots \\ V_{n-1} \end{pmatrix} = \mathcal{E} \begin{pmatrix} -c_1 \\ \sigma_{12} \\ \sigma_{13} \\ \vdots \\ \sigma_{1n-1} \end{pmatrix}. \tag{17}$$

As we mentioned above, only $(n-1)$ equations in (14) are independent, so we skipped the last one. We can now apply Cramer's rule to find I_{out}

$$I_{\text{out}} = \mathcal{E}\frac{\begin{vmatrix} -c_1 & -\sigma_{12} & -\sigma_{13} & \dots & -\sigma_{1n-1} \\ \sigma_{12} & c_2 & -\sigma_{23} & \dots & -\sigma_{2n-1} \\ \sigma_{12} & -\sigma_{32} & c_3 & \dots & -\sigma_{3n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{12} & -\sigma_{n-12} & -\sigma_{n-13} & \dots & c_{n-1} \end{vmatrix}}{\begin{vmatrix} -1 & -\sigma_{12} & -\sigma_{13} & \dots & -\sigma_{1n-1} \\ 0 & c_2 & -\sigma_{23} & \dots & -\sigma_{2n-1} \\ 0 & -\sigma_{32} & c_3 & \dots & -\sigma_{3n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\sigma_{n-12} & -\sigma_{n-13} & \dots & c_{n-1} \end{vmatrix}} = \mathcal{E}\frac{-\det\Sigma'}{-\det\Sigma''}. \tag{18}$$

Therefore, the equivalent conductance reads

$$\sigma_{\text{eq}} = \frac{\det\Sigma'}{\det\Sigma''}. \tag{19}$$

As before, the answer is a ratio of two polynomials of degree $(n-1)$ and $(n-2)$, which clearly has correct units. Each determinant is of the form similar to that of Sec. 1.3. They both are non-zero (positive) for any connected circuit. See Appendix A for more detail. Eq.(19) constitutes the main result of this paper. We discuss its properties in the following section.

3 Discussion

We have derived a closed formula for the equivalent conductance of an arbitrary circuit. All one needs to know is the edge conductances σ_{ij} which give rise to the Σ-matrix defined in (14). The equivalent conductance can then be computed as a ratio of two subdeterminants of Σ via Eq. (19).

One important feature of (19) is that it respects the permutations symmetry. Indeed, relabeling any nodes other than the two connected to the battery terminals (1 and n) must not affect the equivalent conductance of the circuit. For example, switching labels i and j would merely result in a minus sign in front of $\det\Sigma'$ and $\det\Sigma''$ thus keeping the answer intact.

On physical grounds, the equivalent conductance of an arbitrary ciruit (with $\sigma_{ij} < \infty$) can be zero (for a disconnected circuit), but never infinite.

One the other hand, the denominator in Eq. (19) can vanish if suffieciently many σ_{ij} equal zero. This implies that if $\det \Sigma'' = 0$, the other determinant, $\det \Sigma'$, must vanish as well.

Remarkably, both determinants may equal zero, even if the equivalent conductance is finite. For instance, if the conductances in a three-node circuit are $\sigma_{12} = \sigma_{23} = 0 \neq \sigma_{13}$. It is easy to see that in this case $\sigma_{\rm eq} = \sigma_{13}$. So in order for the formula (19) to be well-behaved, one might need to consider the following limiting procedure (akin footnote on p.8). Give the zero σ's small non-zero values and take those values to zero. Again, from a physics point of view, such a procedure should have a well-defined limit.

In Sec. 1.3 we saw that a special arrangement of some edge conductances (1, 2, 4 and 5) resulted in zero current through the middle edge (3). In that case, it is easy to see that $\sigma_{\rm eq}$ does not depend on σ_3. Once such a symmetry is recognized, one can do two things without affecting the equivalent conductance:

- Throw the middle edge away, i.e. set $\sigma_3 = 0$. This can be done, since there is no current through this edge.

- Short-circuit the top and bottom nodes, i.e. put $\sigma_3 \to \infty$. This can be done, since the nodal potentials V_2 and V_3 are equal.

In both cases, the resulting circuit can be easily simplified and the equivalent conductance can be computed according to Eqs. (5) and (6). Importanly, these two resulting circuits are *different*, but have the same equivalent conductance. This may not be as obvious for a more complicated circuit. Suppose there is special edge (with conductance σ_*) such that performing the two operations above yields the same equivalent conductance. We can prove then that $\sigma_{\rm eq}$ does not depend on σ_*, as follows.

As shown in Appendix A, both $\det \Sigma'$ and $\det \Sigma''$ are linear functions of edge conductances. Thus the equivalent conductance can be written as

$$\sigma_{\rm eq} = \frac{A\sigma_* + B}{C\sigma_* + D}.$$

Requiring that $\sigma_{\rm eq}(0) = \sigma_{\rm eq}(\infty)$ yields $A/C = B/D$, which implies that σ_* drops out from the equivalent conductance.

In this paper we focused only on resistive circuits with a single battery. However, the same analysis can be applied to circuits with multiple batteries. This can be done by simply incorporating the additional EMF's into the nodal potential differences. Clearly, ideal batteries would not affect the equivalent resistance between fixed nodes.

Generalization to capacitive circuits is also straightforward, as equvalent capacitance obeys the same rules (5) and (6) as conductance. Thus the final formula (19) can be understood in terms of capacitance as well.

In addition, Eq.(19) will work for the equivalent impedance (admittance) of an AC-circuit. The only difference would be that admittance is a complex number and either $\det\Sigma'$ or $\det\Sigma''$ can be zero even for a connected AC-circuit. In fact, setting the determinants to zero one can determine the resonance frequencies of the circuit.

We conclude by pointing out that the main result of this paper, Eq. (19), unveils a curious interplay between electrical circuits, matrix algebra, graph theory and its applications to computer science. Specifically, there is a straighforward correspondence between electrical circuits and random walks on graphs [2], including the concept of *escape probability*, which is a direct analog of equivalent resistance. In addition, Eq. (19) can help to investigate the connectivity of generic graphs, which is done in e.g. [3] using spectral analysis. These connections are particularly useful, as there is much physical intuition about electrical circuits that could give rise to some less obvious mathematical statements.

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A Properties of the determinants

In general, determinants contain both positive and negative monomials. However, in Sec. 1.3 we saw that, after some cancellations, the determinants in the numerator and denominator of (12) had only positive terms. In this appendix, we shall prove that the determinant of Σ' is always of this form, that is

Lemma 3 *Each term in* $\det\Sigma'$ *enters with a plus.*

Proof: We shall proceed with the proof by induction in n. For example, for $n = 3$,

$$\Sigma' = \begin{pmatrix} c_1 & -\sigma_{12} \\ -\sigma_{21} & c_2 \end{pmatrix},$$

where $c_1 = \sigma_{12} + \sigma_{13}$, $c_2 = \sigma_{21} + \sigma_{23}$ and $\sigma_{12} = \sigma_{21}$. Thus the determinant

$$\det \Sigma' = c_1 c_2 - \sigma_{12}^2 = \sigma_{12}\sigma_{23} + \sigma_{12}\sigma_{13} + \sigma_{13}\sigma_{23}.$$

In addition, in Sec. 1.3 we saw the statement of the Lemma to be true for $n = 4$.

Assume now that the statement holds for $n = k$. In order to prove that it also is true for $n = k+1$ (Σ' is then $k \times k$), it is sufficient to demonstrate that the coefficient in front of each σ_{ij}, as it enters $\det \Sigma'$, is positive. Without loss of generality, we can focus on $\sigma_{12} \equiv \sigma_{21}$. In the determinant

$$\det \Sigma' = \begin{vmatrix} c_1 & -\sigma_{12} & -\sigma_{13} & \dots & -\sigma_{1,k} \\ -\sigma_{21} & c_2 & -\sigma_{23} & \dots & -\sigma_{2,k} \\ -\sigma_{31} & -\sigma_{32} & c_3 & \dots & -\sigma_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_{k,1} & -\sigma_{k,2} & -\sigma_{k,3} & \dots & c_k \end{vmatrix},$$

σ_{12} appears in the four upper-left entries, including inside c_1 and c_2, but nowhere else in the rest of the matrix. Using this fact and properties of determinants, we can rewrite the expression so that σ_{12} will only appear in one place. Specifically we can replace the second row with the sum of itself and the first one and then repeat this procedure with the same columns

$$\begin{aligned} \det \Sigma' &= \begin{vmatrix} c_1 & -\sigma_{12} & -\sigma_{13} & \dots & -\sigma_{1,k} \\ c_1' & c_2' & -(\sigma_{13} + \sigma_{23}) & \dots & -(\sigma_{1,k} + \sigma_{2,k}) \\ -\sigma_{31} & -\sigma_{32} & c_3 & \dots & -\sigma_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_{k,1} & -\sigma_{k,2} & -\sigma_{k,3} & \dots & c_k \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_1' & -\sigma_{13} & \dots & -\sigma_{1,k} \\ c_1' & c_1' + c_2' & -(\sigma_{13} + \sigma_{23}) & \dots & -(\sigma_{1,k} + \sigma_{2,k}) \\ -\sigma_{31} & -(\sigma_{31} + \sigma_{32}) & c_3 & \dots & -\sigma_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_{k,1} & -(\sigma_{k,1} + \sigma_{k,2}) & -\sigma_{k,3} & \dots & c_k \end{vmatrix}. \end{aligned}$$

Here $c_1' = c_1 - \sigma_{12}$ and $c_2' = c_2 - \sigma_{12}$ do not contain σ_{12}. Hence the only entry that depends on σ_{12} is c_1, and it does so linearly. We can expand the latter determinant as

$$\begin{aligned} \det \Sigma' &= \sigma_{12} \begin{vmatrix} c_2' & -(\sigma_{13}+\sigma_{23}) & \dots & -(\sigma_{1,k}+\sigma_{2,k}) \\ -(\sigma_{31}+\sigma_{32}) & c_3 & \dots & -\sigma_{3,k} \\ \vdots & \vdots & \ddots & \vdots \\ -(\sigma_{k,1}+\sigma_{k,2}) & -\sigma_{k,3} & \dots & c_k \end{vmatrix} \\ &+ \text{(terms not containing } \sigma_{12}). \end{aligned} \tag{20}$$

Define $\sigma_{i,2}' := \sigma_{i,1} + \sigma_{i,2}$ for $i = 3, ..., k$. Then the determinant multiplying σ_{12} in Eq. (20) will be of the same form as the original determinant of Σ'. The size of this determinant is $(k-1) \times (k-1)$ (which corresponds to $n = k$), so by the induction hypothesis it must be positive. Since the choice of σ_{12} was arbitrary we have proved that the coefficient in front of each edge conductance in Σ' is positive. Therefore Σ', as a polynomial in σ's, has only positive terms, which proves the lemma. □

Moreover, as Σ'' has a form very similar to that of Σ', the proof above would work for its determinant as well.

B Simplifiable circuits

In this section we consider circuits that contain elements in series or in parallel, as well as circuits that can be reduced for symmetry reasons. Specifically, we are interested in the form of the conductance matrix σ_{ij} and the corresponding Σ-matrix.

B.1 Connection in series

If there is a pair of edges in series, the node (k) shared by these edges would be two-valent and the corresponding row/column in σ_{ij} would have only two non-zero entries, say, σ_{kl} and σ_{km}. This also implies that the nodes l and m are not connected directly, i.e. $\sigma_{lm} = 0$. From physics we know that the two edges can be replaced by one with the equivalent conductance given by (6). Therefore, we can reduce the size of the conductance matrix by crossing out the k^{th} row and column and by setting $\sigma_{lm} \equiv \sigma_{ml} := \left(\sigma_{kl}^{-1} + \sigma_{km}^{-1}\right)^{-1}$.

It is also insightful to investigate this statement mathematically. Without loss of generality, we can set $k = 2$, $l = 3$, and $m = 4$. Then the (top-left part of the) Σ'-matrix for such a circuit would look like

$$\Sigma' = \begin{pmatrix} c_1 & 0 & -\sigma_{13} & -\sigma_{14} & -\sigma_{15} & \cdots \\ 0 & c_2 & -\sigma_{23} & -\sigma_{24} & 0 & \cdots \\ -\sigma_{31} & -\sigma_{32} & c_3 & 0 & -\sigma_{35} & \cdots \\ -\sigma_{41} & -\sigma_{42} & 0 & c_4 & -\sigma_{45} & \cdots \\ -\sigma_{51} & 0 & -\sigma_{52} & -\sigma_{53} & c_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

As explained above, $\sigma_{34} = \sigma_{43} = 0$. Since node 2 is only connected to 3 and 4, the non-zero entries of the second row and column are $\sigma_{23} = \sigma_{32}$, $\sigma_{24} = \sigma_{42}$ and $c_2 = \sigma_{23} + \sigma_{24}$. On the other hand, the matrix describing the reduced curcuit is

$$\tilde{\Sigma}' = \begin{pmatrix} c_1 & -\sigma_{13} & -\sigma_{14} & -\sigma_{15} & \cdots \\ -\sigma_{31} & \tilde{c}_3 & \tilde{\sigma}_{34} & -\sigma_{35} & \cdots \\ -\sigma_{41} & \tilde{\sigma}_{43} & \tilde{c}_4 & -\sigma_{45} & \cdots \\ -\sigma_{51} & -\sigma_{53} & -\sigma_{54} & c_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{22}$$

Here $\tilde{\sigma}_{34} = \left(\sigma_{23}^{-1} + \sigma_{24}^{-1}\right)^{-1} \equiv \sigma_{23}\sigma_{24}/(\sigma_{23} + \sigma_{24})$, whereas $\tilde{c}_3$ and $\tilde{c}_4$ include $\tilde{\sigma}_{34} = \tilde{\sigma}_{43}$. We can similarly introduce the submatrices Σ'' and $\tilde{\Sigma}''$. We shall now prove the following

Lemma 4

$$\sigma_{\text{eq}} = \frac{\det \Sigma'}{\det \Sigma''} = \frac{\det \tilde{\Sigma}'}{\det \tilde{\Sigma}''} = \tilde{\sigma}_{\text{eq}} \tag{23}$$

Proof: In what follows, for the sake of compactness we omit the dots in the determinants. We start by expanding $\det \Sigma'$ from (21) in the elements of the second row

$$\begin{aligned} \det \Sigma' &= c_2 \begin{vmatrix} c_1 & -\sigma_{13} & -\sigma_{14} & -\sigma_{15} \\ -\sigma_{31} & c_3 & 0 & -\sigma_{35} \\ -\sigma_{41} & 0 & c_4 & -\sigma_{45} \\ -\sigma_{51} & -\sigma_{53} & -\sigma_{54} & c_5 \end{vmatrix} \\ &+ \sigma_{23} \begin{vmatrix} c_1 & 0 & -\sigma_{14} & -\sigma_{15} \\ -\sigma_{31} & -\sigma_{32} & 0 & -\sigma_{35} \\ -\sigma_{41} & -\sigma_{42} & c_4 & -\sigma_{45} \\ -\sigma_{51} & 0 & -\sigma_{54} & c_5 \end{vmatrix} + \sigma_{24} \begin{vmatrix} c_1 & -\sigma_{13} & 0 & -\sigma_{15} \\ -\sigma_{31} & c_3 & -\sigma_{32} & -\sigma_{35} \\ -\sigma_{41} & 0 & c_4 & -\sigma_{42} \\ -\sigma_{51} & -\sigma_{53} & 0 & c_5 \end{vmatrix}. \end{aligned} \tag{24}$$

Note that we switched columns 2 and 3 in the last determinant. Using that $c_2 = \sigma_{23} + \sigma_{34}$, we can collect similar terms as

$$\det \Sigma' = \sigma_{23} \begin{vmatrix} c_1 & -\sigma_{13} & -\sigma_{14} & -\sigma_{15} \\ -\sigma_{31} & c_3 - \sigma_{32} & 0 & -\sigma_{35} \\ -\sigma_{41} & -\sigma_{42} & c_4 & -\sigma_{45} \\ -\sigma_{51} & -\sigma_{53} & -\sigma_{54} & c_5 \end{vmatrix} + \sigma_{24} \begin{vmatrix} c_1 & -\sigma_{13} & -\sigma_{14} & -\sigma_{15} \\ -\sigma_{31} & c_3 & -\sigma_{32} & -\sigma_{35} \\ -\sigma_{41} & 0 & c_4 - \sigma_{42} & -\sigma_{45} \\ -\sigma_{51} & -\sigma_{53} & -\sigma_{54} & c_5 \end{vmatrix}.$$

It is now easy to see that the sum of columns 2 and 3 is the same for both determinants. Thus we can replace column 2 in each term with this sum, which would allow us to combine the two determinants into one

$$\begin{vmatrix} c_1 & -(\sigma_{13} + \sigma_{14}) & -(\sigma_{23} + \sigma_{24})\sigma_{14} & -\sigma_{15} \\ -\sigma_{31} & c_3 - \sigma_{23} & \sigma_{23}\sigma_{24} & -\sigma_{35} \\ -\sigma_{41} & c_4 - \sigma_{42} & (\sigma_{23} + \sigma_{24})c_4 - \sigma_{24}^2 & -\sigma_{45} \\ -\sigma_{51} & -(\sigma_{53} + \sigma_{54}) & -(\sigma_{23} + \sigma_{24})\sigma_{54} & c_5 \end{vmatrix}.$$

We can now factor $(\sigma_{23}+\sigma_{24})$ out of the third column and use $(\sigma_{23}\sigma_{24})/(\sigma_{23}+\sigma_{24}) \equiv \tilde{\sigma}_{34}$ (in the second entry) together with $\sigma_{24}^2/(\sigma_{23}+\sigma_{24}) \equiv \sigma_{24} - \tilde{\sigma}_{34}$ (in the third entry) to obtain

$$(\sigma_{23} + \sigma_{24}) \begin{vmatrix} c_1 & -(\sigma_{13} + \sigma_{14}) & -\sigma_{14} & -\sigma_{15} \\ -\sigma_{31} & c_3 - \sigma_{23} & \tilde{\sigma}_{34} & -\sigma_{35} \\ -\sigma_{41} & c_4 - \sigma_{42} & c_4 - \sigma_{24} + \tilde{\sigma}_{34} & -\sigma_{45} \\ -\sigma_{51} & -(\sigma_{53} + \sigma_{54}) & -\sigma_{54} & c_5 \end{vmatrix},$$

where the third diagonal entry is precisely $c_4 - \sigma_{24} + \tilde{\sigma}_{34} \equiv \tilde{c}_4$. Similarly $c_3 - \sigma_{23} + \tilde{\sigma}_{34} \equiv \tilde{c}_3$. With that in mind, subtracting the third column from the second one yields

$$\det \Sigma' = (\sigma_{23} + \sigma_{24}) \begin{vmatrix} c_1 & -\sigma_{13} & -\sigma_{14} & -\sigma_{15} \\ -\sigma_{31} & \tilde{c}_3 & -\tilde{\sigma}_{34} & -\sigma_{35} \\ -\sigma_{41} & -\tilde{\sigma}_{34} & \tilde{c}_4 & -\sigma_{45} \\ -\sigma_{51} & -\sigma_{53} & -\sigma_{54} & c_5 \end{vmatrix} \equiv (\sigma_{23} + \sigma_{24}) \det \tilde{\Sigma}'.$$

Repeating the same consideration without the first row and column we would get

$$\det \tilde{\Sigma}'' = (\sigma_{23} + \sigma_{24}) \begin{vmatrix} \tilde{c}_3 & -\tilde{\sigma}_{34} & -\sigma_{35} \\ -\tilde{\sigma}_{34} & \tilde{c}_4 & -\sigma_{45} \\ -\sigma_{53} & -\sigma_{54} & c_5 \end{vmatrix} \equiv (\sigma_{23} + \sigma_{24}) \det \tilde{\Sigma}'',$$

from which (23) follows. $\square$

B.2 Connection in parallel

If two nodes are connected by multiple edges, those edges can be replaced by one with the total conductance from (5). This is the reason we did not consider mulitple edges from the start, without any loss of generality.

B.3 Short-circuiting two nodes

The two above simplifications do not affect the equivalent conductance of the whole circuit. A more interesting scenario (that does, in general) is short-circuiting two nodes, say i and j, by connecting them with an ideal wire. Mathematically this corresponds to letting $\sigma_{ij} \to \infty$, whereas the physical implication is the equality of the corresponding nodal potentials $V_i = V_j$. Such nodes can be simpy merged together, becoming one. After the merging, some edges that were not parallel may become parallel. Thus we can use the idea of the previous paragraph.

In essence, this short-circuiting eliminates one unknown potential, hence reduces the size of the conductance matrix. Instead of the two rows/columns (i and j) we have only one. The entries of this new row/column are given by

$$\sigma'_{ik} = \sigma_{ik} + \sigma_{jk}, \tag{25}$$

for any $k \neq i, j$.

So far we have not discussed the diagonal elements of σ_{ij}. One reason being is that they do not affect the Σ-matrix, which has been most relevant in this paper. In fact, we can generalize the procedure of going from σ_{ij} to Σ as follows

$$\Sigma_{ij} = \mathrm{Diag}(c_1, c_2, ..., c_n) - \sigma_{ij}, \tag{26}$$

with the same c's as before: $c_i = \sum_{j=1}^{n} \sigma_{ij}$. Clearly, no matter what σ_{ii} are, they drop out from Σ, according to (26).

At the same time, what would be the meaning of σ_{ii}? How can a node be connected to itself? Now, in the spirit of the short-circuiting scenario above, one can think of each node as "self short-circuited". In other words, $\sigma_{ii} = \infty$. In fact, this observation will make the short-circuiting recipe (25) valid for diagonal elements as well. Notice that all of the manipulations with σ_{ij} discussed in this appendix work in the same exact way for the Σ-matrix. In other words, these manipulations "commute" with (26).

To conclude, we would like to comment of the symmetry issue. Some nodal potentials may turn out to be equal on symmetry grounds, even if the corresponding nodes are not short-circuited. For instance, as we pointed out in Section 3, the Wheatstone bridge circuit has $V_2 = V_3$, if (13) is satisfied. Another example is the classic cube-circuit problem, where there are triples of (not connected) nodes having the same potential. Importantly, in these symmetric situations, the short-circuiting does not affect the equivalent conductance. As such symmetries are not as manifest in more complicated circuits, it would be interesting to come up with a way of detecting the equipotential nodes by looking at the form of the Σ matrix.

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